TURING AND WITTGENSTEIN ON LOGIC AND MATHEMATICS

THE EIGHTEENTH BRITISH WITTGENSTEIN SOCIETY LECTURE
RAY MONK
Alan Turing (1912-1954)

1931-34 undergraduate student of mathematics at Cambridge
1935 aged just 22, made a Fellow of King’s College, Cambridge
1936 publishes “On Computable Numbers, with an Application to the Entscheidungsproblem”
1936-38 studies with Alonzo Church at Princeton
1938-39 lectures at Cambridge
1939-45 takes a leading part in the codebreaking work at Bletchley Park
1945-47 pioneers development of digital computers
1948-54 Reader in Mathematics and Deputy Director of Computing Machine Laboratory, Manchester
1950 publishes ‘Computing Machinery and Intelligence’
1952 convicted of ‘gross indecency’
1954 dies of cyanide poisoning, possibly suicide
In December 1933, during his final year as an undergraduate and the academic year in which Wittgenstein dictated the *Blue Book*, Turing gave a paper on ‘Mathematics and Logic’ to the Moral Sciences Club in Cambridge. The text of the talk has not survived, but the minutes of the meeting record that Turing ‘suggested that a purely logistic view of mathematics was inadequate and that mathematical propositions possessed a variety of interpretations of which the logistic was merely one.’
The Entscheidungsproblem (Decision problem)

Given any statement $p$ in a system $S$, is there an effective method (an algorithm) for determining whether or not $p$ is provable in $S$?

According to Hilbert, this was ‘the main problem of mathematical logic’.
The Entscheidungsproblem is ‘one of the leading problems of mathematical logic’
(Frank Ramsey, ‘On a Problem in Formal Logic’, written in 1928, published posthumously in 1930)

‘To that I would reply: Such “leading problems” cannot exist!’
(Wittgenstein in conversation with the Vienna Circle)

‘A “leading problem of mathematical logic” is for us a problem of mathematics like any other’ (Philosophical Investigations, §124)
‘On Computable Numbers, with an Application to the Entscheidungsproblem’ (1936)

‘Corresponding to each computing machine M we construct a formula Un(M) and we show that, if there is a general method for determining whether Un(M) is provable, then there is a general method for determining whether M ever prints 0’

‘We can show further that there can be no machine E which, when supplied with the S.D ["program"] of an arbitrary machine M, will determine whether M ever prints a given symbol (0 say)’

So, the answer to Hilbert’s question is: no, there is no effective method for determining whether any given statement is or is not provable in a system.
In February 1937, Turing asked for a copy of ‘On Computable Numbers’ to be sent to Wittgenstein, and a few months later the two were introduced to each other personally by their mutual friend Alistair Watson, a mathematician who had been a Fellow of King’s since 1933. After their first meeting, Turing and Wittgenstein would sometimes meet, together with Watson, in the Botanic Garden in Cambridge to discuss the foundations of mathematics.
In 1938, Watson published an article in *Mind* on the foundations of mathematics, which contains a discussion of Gödel’s incompleteness theorems that, he tells the reader, ‘owes much to lengthy discussions with a number of people, especially Mr. Turing and Dr. Wittgenstein of Cambridge.’ The view it presents is close to the anti-foundationalist view associated with Wittgenstein. Watson is sceptical that the ‘subject known as the Foundations of Mathematics’ has made any progress in providing ‘the justification of valid mathematical arguments’ or ‘described correctly the status of mathematics’. These new ‘schools of thought’, he insists, ‘make no advance over long-known philosophical theories, and do not provide a solution of the ancient problems’. His article targets, in particular, the logicism of Russell. What Watson, Turing and Wittgenstein seem to have had in common, then, is that none of them accepted Russell’s logicism. Watson and Wittgenstein also shared the view that the branch of mathematics called the ‘foundations of mathematics’ does not, in fact, provide the foundations of mathematics. Where did Turing stand on this issue? Juliet Floyd thinks he agreed with Watson and Wittgenstein. I do not.
But this is surely over-stating the similarities:

‘Beginning with a Moral Sciences Club talk in 1933, Turing developed an “anthropological” approach to the foundations of logic, influenced by Wittgenstein, in which “common sense” plays a foundational role.’

‘Turing’s “On Computable Numbers” provides, one might say, a kind of ex post facto justification for Wittgenstein’s post-1933 “language game” approach to definability and to logic: a piecemeal, step-by-step exploration of possible meanings, processes, rules, using simplified snapshots of an evolving series of possible routines and modes of argumentation, portrayed as embedded and embodied in an evolving environment and culture. “Logic” is on this view fundamentally erected in the course of the investigation, carried forward in pieces of technology embedded in everyday language, and not by devising a particular formal system or finding any particular bottom level of analysis or uncontroversial starting point.’

(Juliet Floyd, ‘Turing on “Common Sense”: Cambridge Resonances’)
‘The notion of tautology was Wittgenstein’s central contribution to logic and philosophy, as Turing knew. The whole argument of “On Computable Numbers” is a practical realization – and profound generalization – of an idea Wittgenstein had injected into the heart of Cambridge discussion. The fact that in our language we can put together declarative sayings that obey all the ordinary rules of grammar and yet “cancel out” their saying anything shows us something important in being empty, something important about logic and its limits. This perspective – as Russell noted at the end of Introduction to Mathematical Philosophy – transforms the idea of analytic or self-evident “truth”, sinking it into logical activity itself, now regarded as a manifestation of our ability to appreciate limits in the totality of what we can sensibly say, represent, or derive. The role of such limiting cases of expression may be clarified by reflecting on tautologies in a suitable symbolism, just as Wittgenstein had done in the Tractatus, and as Turing also did, far more generally and rigorously, in “On Computable Numbers”.’ (Juliet Floyd, ‘Turing and “Common Sense”: Cambridge Resonances)
‘Solvable and Unsolvable Problems’

In this article, Turing uses results in mathematical logic, most notably his own result in ‘On Computable Numbers’, to show (a) that there are problems that are unsolvable by algorithmic means, and (b) that there is no systematic procedure for determining whether a problem is solvable or not. At the end of the article, he claims that these results set ‘certain bounds to what we can hope to achieve purely by reasoning’. He adds:

‘These, and some other results of mathematical logic may be regarded as going some way towards a demonstration within mathematics itself, of the inadequacy of “reason” unsupported by common sense.’

In its appeal to results in mathematical logic and its conception of ‘reason’, this article seems to me fundamentally non-anthropological and un-Wittgensteinian.
The intellectual gap between Turing and Wittgenstein: the relationship between logic and mathematics

1939, both Wittgenstein and Turing give lectures at Cambridge under the title ‘The Foundations of Mathematics’. They could not have been more different.

‘The Turing course was one on the chess game of mathematical logic; extracting the neatest and tightest set of axioms from which to begin, making them flower according to the exact system of rules into the structures of mathematics, and discovering the technical limitations of that procedure.’
(Andrew Hodges, Alan Turing: The Enigma)
Turing: ‘You seem to be saying that if one uses a little common sense, one will not get into trouble.’

Wittgenstein: ‘No, that is NOT what I mean at all.’
‘What would go wrong, if anything, if we didn’t recognize the law of contradiction? – or any other proposition in Russell’s logic?’ (Lecture XIX)

‘Think of the case of the Liar. It is very queer in a way that this should have puzzled anyone … Because the thing works like this: if a man says “I am lying” was say that it follows that he is not lying, from which it follows that he is lying and so on. Well, so what? You can go on like that until you are black in the face. Why not? It doesn’t matter…

Now suppose a man says “I am lying” and I say “Therefore you are not, therefore you are, therefore you are not…” – What is wrong? Nothing. Except that it is no use; it is just a useless language-game, and why should anybody be excited?’
(Lecture XXI)
Turing: What puzzles one is that one usually uses a contradiction as a criterion for having done something wrong. But in this case one cannot find anything done wrong.

Wittgenstein: Yes – and more: nothing has been done wrong.

Wittgenstein: There is a particular mathematical method, the method of reduction ad absurdum, which we might call “avoiding the contradiction”. In this method one shows a contradiction and then shows the way from it. But this doesn’t mean that a contradiction is a sort of devil. One may say, “From a contradiction everything would follow.” The reply to that is: Well then, don’t draw any conclusions from a contradiction; make that the rule. You might put it: There is always time to deal with a contradiction when we get to it. When we get to it, shouldn’t we simply say, “This is no use – and we won’t draw any conclusions from it”?

Turing: The real harm will not come in unless there is an application, in which case a bridge may fall down or something of that sort.
Wittgenstein: It was suggested last time that the danger with a contradiction in logic or mathematics is in the application. Turing suggested that a bridge might collapse. Now it does not sound quite right to say that a bridge might fall down because of a contradiction. We have an idea of the sort of mistake which would lead to a bridge falling.
(a) We’ve got hold of a wrong natural law – a wrong coefficient.
(b) There has been a mistake in calculation – someone has multiplied wrongly.
The first case obviously has nothing to do with having a contradiction; and the second is not quite clear.
Turing: You cannot be confident about applying your calculus until you know that there is no hidden contradiction in it.

Wittgenstein: There seems to me to be an enormous mistake there. For your calculus gives certain results, and you want the bridge not to break down. I’d say things can go wrong is only two ways: either the bridge breaks down or you have made a mistake in your calculation – for example, you multiplied wrongly. But you seem to think that there may be a third thing wrong: the calculus is wrong.

Turing: No. What I object to is the bridge falling down.

Wittgenstein: But how do you know that it will fall down? Isn’t that a question of physics?

Turing: If one takes Frege’s symbolism and gives someone the technique of multiplying in it, then by using a Russell paradox he could get a wrong multiplication.

Wittgenstein: This would come to doing something which we would not call multiplying... The point I’m driving at is that Frege and Russell’s logic is not the basis for arithmetic anyway – contradiction or no contradiction. (from Lectures XXII and XXIII)
17 December 1930, Wittgenstein told the Circle he had been reading ‘a work by Hilbert on consistency’ (probably ‘Neubegründung der Mathematik’ (‘A New Foundation for Mathematics’). His reaction to Hilbert’s concern with proving consistency was to ask: ‘Is it even possible for mathematics to be inconsistent? I would like to ask these people. What are you really up to? Do you really believe that there are contradictions hidden in mathematics?’

Following Frege, Wittgenstein distinguished two roles for the axioms of a formal system: 1. Specifying the rules of the game, and 2. Laying forth the opening positions of a game. If understood in the second way, Wittgenstein says, then contradictory axioms would represent just one among many configurations and ‘what is there to get excited about if this configuration turns up?’ Understood as rules, too, there is nothing to get excited about. If we find that two rules contradict one another, we should simply ‘introduce a new rule and the conflict is resolved.’
28 December 1930.

‘These antinomies [the paradoxes] have nothing whatsoever to do with the consistency of mathematics; there is no connection here at all.

For the antinomies did not arise in the calculus but in our ordinary language, precisely because we use words ambiguously. Hence the resolution of the antinomies consists in replacing the hazy way of expressing ourselves by a precise one (by recalling the real meanings of our words). Thus the antinomies vanish by means of an analysis, not by means of proof.

If the contradictions in mathematics arise through an unclarity, then I can never dispel this unclarity by a proof. A proof proves only what it does prove. But it cannot lift the fog. What is needed is an analysis, not a proof. A proof cannot dispel the fog.

This is enough to show that there can be no such thing as a proof of consistency (if you imagine the contradictions of mathematics as being of the same kind as the contradictions of set theory), that this proof cannot accomplish what it is expected to do.

If I am unclear about the nature of mathematics, no proof can help me. And if I am clear about the nature of mathematics, then the question about its consistency cannot arise at all.’
Wittgenstein’s last mention of Turing

‘You’re quite right, a mathematician by name of Turing attended my lectures in ’39 (they were pretty poor) & it’s probably the same man who wrote the article you mention. I haven’t read it but I imagine it’s no leg pull.’

(Wittgenstein to Norman Malcolm, 1 December 1950)
Wittgenstein died of cancer, 29 April 1951, aged 62. His last words were: ‘Tell them I’ve had a wonderful life’

Turing died of cyanide poisoning, 7 June 1954, aged 41. A half-eaten apple lay beside his bed. This, it is said, inspired the logo of Apple Inc.